

MIDTERM TEST

Semester 1, Academic year 2011-2012

Duration: 90 minutes

SUBJECT: Calculus 1	
Head of Department of Mathematics	Lecturers: Prof. P.Q. Khanh, Prof. N.V. Thu
Professor Phan Quoc Khanh	Dr. P.H.A. Ngoc, Dr. T.T. Duong
Signature:	Signature:

INSTRUCTIONS: Open-book examination. Computers and laptops prohibited. Exchanging documents strictly prohibited. Students answer the questions in Part II directly on the question sheets.

PART I: Long answer questions

Question 1. (20 marks) Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither?

$$f(x) := \begin{cases} x^2 + 1 & \text{if } x < 0 \\ e^{x-1} & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } x > 1. \end{cases}$$

Question 2. (10 marks) Prove that the equation

$$e^{-x^2+1} = x + 1,$$

has a root in the interval $(0, \infty)$.

Question 3. (15 marks) Find $\frac{dy}{dx}$ by implicit differentiation.

$$x^2 \cos y + \sin 2y = xy.$$

Question 4. (15 marks) Use logarithmic differentiation to find the derivative of the function

$$y(x) = \sqrt{x}e^{x^2}(x^2 + 1)^{2011}.$$

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PART II. Multi-choice questions

In each of the following questions, choose the best answer a), b), c), d), or e). Each question carries 10 points.

Question 5. Evaluate

$$\lim_{h \rightarrow 25} \frac{\sqrt{h} - 5}{h - 25}$$

a. 0	b. $\frac{1}{25}$	c. $\frac{1}{10}$	d. $\frac{1}{5}$	e. nonexistent
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Question 6. Find $\lim_{x \rightarrow 0} f(x)$ where

$$f(x) = \begin{cases} 4 - x & x < 0 \\ 2 & x = 0 \\ x + 1 & x > 0 \end{cases}$$

a. 4	b. 2	c. 1	d. 0	e. nonexistent
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Question 7. Let $f(x) = x \arcsin x$. Find $f'(x)$

a. $\frac{x}{\sqrt{1-x^2}}$	b. $\arcsin x - x(\arcsin x)^{-2}$	c. $\frac{x}{\sqrt{1-x^2}} + \arcsin x$	d. $\frac{x}{\sqrt{1-x^2}} - \arcsin x$	e. $\frac{1}{\sqrt{1-x^2}}$
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Question 8. Suppose f and f^{-1} are both differentiable for all x with $f(3) = 5$, $f'(3) = 7$. Which of the following must be a line tangent to the graph of f^{-1} ?

a. $y = 5 + 7(x - 3)$	b. $y = \frac{1}{5} + \frac{1}{7}(x - 3)$	c. $y = 3 + 7(x - 5)$	d. $y = \frac{1}{3} + \frac{1}{7}(x - 5)$
e. $y = 3 + \frac{1}{7}(x - 5)$			

END.

ANSWERS OF MIDTERM TEST

PART I: Long answer questions

1) Note that

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} (x^2 + 1) = 1 \neq f(0) = e^{-1},$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} e^{x-1} = e^{-1} = f(0),$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} e^{x-1} = 1 = f(1),$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} (3 - x) = 2 \neq f(1).$$

Thus, the function f is discontinuous at 0 and 1. However, it is continuous from the right at 0 and from the left at 1.

2) Consider the function f defined by

$$f(x) = e^{-x^2+1} - x - 1, \quad x \in [0, \infty).$$

It is clear that f is continuous on $[0, \infty)$. Moreover, we have $f(0) = e - 1 > 0$ and $f(2) = e^{-3} - 2 - 1 < 0$. Thus, there exists $c \in (0, 2)$ such that $f(c) = 0$. This implies that the equation $e^{-x^2+1} = x + 1$, has a root in $(0, \infty)$.

3) By implicit differentiation, we have

$$2x \cos y - x^2(\sin y)y' + 2y' \cos 2y = y + xy'.$$

Therefore,

$$y' = \frac{2x \cos y - y}{x^2 \sin y + x - 2 \cos 2y}.$$

4) Take logarithms of both sides of $y(x) = \sqrt{x}e^{x^2}(x^2 + 1)^{2011}$, to get

$$\ln y = \frac{1}{2} \ln x + x^2 + 2011 \ln(x^2 + 1), \quad x > 0.$$

By implicit differentiation, we have

$$\frac{y'}{y} = \frac{1}{2x} + 2x + 2011 \frac{2x}{1 + x^2}, \quad x > 0.$$

$$y' = \sqrt{x}e^{x^2}(x^2 + 1)^{2011} \left(\frac{1}{2x} + 2x + 4022 \frac{x}{1 + x^2} \right), \quad x > 0.$$

PART II. Multi-choice questions

5) c 6) e 7) c 8) e.